On the accuracy of ancestral sequence reconstruction for ultrametric trees with parsimony

Supplementary material

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1 Proof of Lemma 1

To prove Lemma 1 we show that for any rooted binary phylogenetic tree T under a symmetric 4-state substitution model

$$P_\alpha(X) \geq P_\beta(X), \tag{1.1}$$
$$P_{\alpha\beta}(X) \geq P_{\beta\gamma}(X), \tag{1.2}$$
$$P_{\alpha\beta\gamma}(X) \geq P_{\beta\gamma\delta}(X) \tag{1.3}$$

by induction on n. For $n = 2$ the subtrees Y_1 and Y_2 both contain one leaf, which leads to

$$\begin{aligned}
&P_\alpha(X) = (1-3p_1)(1-3p_2),\\
&P_\beta(X) = p_1 p_2,\\
&P_{\alpha\beta}(X) = (1-3p_1)p_2 + p_1(1-3p_2),\\
&P_{\beta\gamma}(X) = 2p_1p_2,\\
&P_{\alpha\beta\gamma}(X) = 0,\\
&P_{\beta\gamma\delta}(X) = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
P_\alpha(X) - P_\beta(X) &= (1-3p_1)(1-3p_2) - p_1p_2 = 1 - 3p_1 - 3p_2 + 9p_1p_2 - p_1p_2 = 1 - 3p_1 - 3p_2 + 8p_1p_2\\
&= \underbrace{(1-4p_1)}_{\geq 0}\underbrace{(1-4p_2)}_{\geq 0} + p_1\underbrace{(1-4p_2)}_{\geq 0} + \underbrace{(1-4p_1)}_{\geq 0}p_2 \geq 0 \text{ as } 0 \leq p_1, p_2 \leq \frac{1}{4}.
\end{aligned}$$

Moreover

$$P_{\alpha\beta}(X) - P_{\beta\gamma}(X) = (1-3p_1)p_2 + p_1(1-3p_2) - 2p_1p_2 = p_1\underbrace{(1-4p_2)}_{\geq 0} + \underbrace{(1-4p_1)}_{\geq 0}p_2 \geq 0 \text{ as } 0 \leq p_1, p_2 \leq \frac{1}{4},$$

and

$$P_{\alpha\beta\gamma}(X) - P_{\beta\gamma\delta}(X) = 0 - 0 = 0 \geq 0,$$

which completes the base case of the induction. For the inductive step we first state some more recursions using (9), (10), (11), (12) and (13):

$$\begin{aligned}
P_\beta(X) = &P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\beta)}(Y_2)\\
&+ 2P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) + 2P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) + 2P_{(\alpha\beta)}(Y_1)P_{(\beta\gamma)}(Y_2)\\
&+ 2P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) + 2P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) + 2P_{(\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2)\\
&+ 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) + P_{(\beta\gamma\delta)}(Y_1)P_{(\beta)}(Y_2)\\
&+ 2P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2)\\
&+ P_{(\beta\gamma\delta)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\beta)}(Y_2),
\end{aligned} \tag{1.4}$$

$$\begin{aligned}P_{\beta\gamma}(X) = {} & P_{(\beta)}(Y_1)P_{(\gamma)}(Y_2) + P_{(\gamma)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\ & + P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) + P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\ & + P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) + P_{(\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) \\ & + P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\beta\gamma)}(Y_2), \end{aligned} \tag{1.5}$$

$$\begin{aligned}P_{\beta\gamma\delta}(X) = {} & 3P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) + 3P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\ & + P_{(\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2). \end{aligned} \tag{1.6}$$

Furthermore, for $i \in \{1,2\}$ we define $P_i := 1 - 4p_i$, and similarly $P := 1 - 4p$.
Moreover, we have that for $i \in \{1,2\}$

$$\begin{aligned}P_{(\alpha)}(Y_i) - P_{(\beta)}(Y_i) &= (1-3p_i)P_\alpha(Y_i) + 3p_iP_\beta(Y_i) - (1-p_i)P_\beta(Y_i) - p_iP_\alpha(Y_i) \\ &\quad \text{by } (8), (9) \\ &= (1-4p_i)P_\alpha(Y_i) - (1-4p_i)P_\beta(Y_i) \\ &= P_i\Big(P_\alpha(Y_i) - P_\beta(Y_i)\Big) \\ &\quad \text{by the definition of } P_i \end{aligned} \tag{1.7}$$

and thus

$$P_{(\alpha)}(Y_i) = P_{(\beta)}(Y_i) + P_i\Big(P_\alpha(Y_i) - P_\beta(Y_i)\Big). \tag{1.8}$$

In the same manner by (10), (11), (12) and (13) we can see that

$$P_{(\alpha\beta)}(Y_i) - P_{(\beta\gamma)}(Y_i) = P_i\Big(P_{\alpha\beta}(Y_i) - P_{\beta\gamma}(Y_i)\Big), \tag{1.9}$$

$$P_{(\alpha\beta\gamma)}(Y_i) - P_{(\beta\gamma\delta)}(Y_i) = P_i\Big(P_{\alpha\beta\gamma}(Y_i) - P_{\beta\gamma\delta}(Y_i)\Big). \tag{1.10}$$

Therefore

$$P_{(\alpha\beta)}(Y_i) = P_{(\beta\gamma)}(Y_i) + P_i\Big(P_{\alpha\beta}(Y_i) - P_{\beta\gamma}(Y_i)\Big), \tag{1.11}$$

$$P_{(\alpha\beta\gamma)}(Y_i) = P_{(\beta\gamma\delta)}(Y_i) + P_i\Big(P_{\alpha\beta\gamma}(Y_i) - P_{\beta\gamma\delta}(Y_i)\Big). \tag{1.12}$$

Additionally we have the following: choose sets A_1, A_2 from $\{\{\alpha\}, \{\alpha\beta\}, \{\alpha\beta\gamma\}\}$ and B_1, B_2 from $\{\{\beta\}, \{\beta\gamma\}, \{\beta\gamma\delta\}\}$ such that for $i \in \{1,2\}$ $|A_i| = |B_i|$, respectively. Then we have that

$$\begin{aligned}& P_{(A_1)}(Y_1)P_{(A_2)}(Y_2) - P_{(B_1)}(Y_1)P_{(B_2)}(Y_2) \\ =& \Big(P_{(B_1)}(Y_1) + P_1\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\Big)\Big(P_{(B_2)}(Y_2) + P_2\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big)\Big) \\ & - P_{(B_1)}(Y_1)P_{(B_2)}(Y_2) \\ & \text{by } (1.8), (1.11) \text{ or } (1.12) \\ =& P_{(B_1)}(Y_1)P_{(B_2)}(Y_2) + P_{(B_1)}(Y_1)P_2\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big) \\ & + P_{(B_2)}(Y_2)P_1\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big) \\ & + P_1P_2\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big) - P_{(B_1)}(Y_1)P_{(B_2)}(Y_2) \\ =& P_{(B_1)}(Y_1)P_2\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big) + P_{(B_2)}(Y_2)P_1\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big) \\ & + P_1P_2\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big). \end{aligned} \tag{1.13}$$

Now suppose that T has n taxa and that (1.1),(1.2) and (1.3) are true for all trees having fewer that n taxa. Note that therefore (1.13) is non-negative, since Y_1 and Y_2 contain both fewer than than n taxa. Then

$$
\begin{aligned}
& P_\alpha(X) - P_\beta(X) \\
=& P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) + 3P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) + 3P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2) \\
& + 3P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 3P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) + 6P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) \\
& + 3P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 3P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) \\
& + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) \\
& - P_{(\alpha\beta)}(Y_1)P_{(\beta)}(Y_2) - 2P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) - 2P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) \\
& - 2P_{(\alpha\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) - 2P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) - 2P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
& - 2P_{(\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
& - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta)}(Y_2) - 2P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
& - P_{(\alpha\beta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) \\
& - P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\beta)}(Y_2) \\
& \text{by (15) and (1.4)} \\
=& P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) \\
& + 2P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - 2P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
& + P_{(\alpha\beta)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\alpha\beta)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
& + 2P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) - 2P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
& + 2P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2) - 2P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) \\
& + 2P_{(\alpha\beta\gamma)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + 2P_{(\alpha\beta\gamma)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
& + P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
& + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta)}(Y_2) \\
& + 2P_{(\alpha\beta)}(Y_2)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big) + 2P_{(\alpha\beta)}(Y_1)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big) \\
& + 2P_{(\alpha\beta\gamma)}(Y_2)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big) + 2P_{(\alpha\beta\gamma)}(Y_1)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big) \\
& + P_{(\alpha\beta)}(Y_2)\Big(P_{(\alpha\beta\gamma)}(Y_1) - P_{(\beta\gamma\delta)}(Y_1)\Big) + P_{(\alpha\beta)}(Y_1)\Big(P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma\delta)}(Y_2)\Big) \\
& + P_{(\alpha\beta\gamma\delta)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\alpha\beta\gamma\delta)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
=& P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) \\
& + 2\Big(P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2)\Big) \\
& + 2\Big(P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2)\Big) \\
& + 2\Big(P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2) - 2P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2)\Big) \\
& + P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
& + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta)}(Y_2) \\
& + \Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_2)\Big) \\
& + \Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1) + 2P_{(\alpha\beta\gamma)}(Y_1) + P_{(\alpha\beta\gamma\delta)}(Y_1)\Big) \\
& + \Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big)\Big(2P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_2)\Big) \\
& + \Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big)\Big(2P_{(\alpha\beta)}(Y_1) + 2P_{(\alpha\beta\gamma)}(Y_1)\Big) \\
& + \Big(P_{(\alpha\beta\gamma)}(Y_1) - P_{(\beta\gamma\delta)}(Y_1)\Big)P_{(\alpha\beta)}(Y_2) + \Big(P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma\delta)}(Y_2)\Big)P_{(\alpha\beta)}(Y_1)
\end{aligned}
$$

$$
\begin{aligned}
=&P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta)}(Y_2)\\
&+ 2\Big(P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2)\Big)\\
&+ 2\Big(P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2)\Big)\\
&+ 2\Big(P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2)\Big)\\
&+ P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2)\\
&+ P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta)}(Y_2)\\
&+ P_1\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+ P_2\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1) + 2P_{(\alpha\beta\gamma)}(Y_1) + P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+ P_1\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big)\Big(2P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_2)\Big)\\
&+ P_2\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big)\Big(2P_{(\alpha\beta)}(Y_1) + 2P_{(\alpha\beta\gamma)}(Y_1)\Big)\\
&+ P_1\Big(P_{\alpha\beta\gamma}(Y_1) - P_{\beta\gamma\delta}(Y_1)\Big)P_{(\alpha\beta)}(Y_2) + P_2\Big(P_{\alpha\beta\delta}(Y_2) - P_{\beta\gamma\delta}(Y_2)\Big)P_{(\alpha\beta)}(Y_1)\\
&\text{by } (1.7), (1.9) \text{ and } (1.10).
\end{aligned}
$$

By (1.13) and the inductive assumption this term is non-negative, and therefore concludes the proof for $P_\alpha(X) \geq P_\beta(X)$. We now proceed with the second part of Lemma 1.

$$
\begin{aligned}
&P_{\alpha\beta}(X) - P_{\beta\gamma}(X)\\
=&P_{(\alpha)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2)\\
&+ 2P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2)\\
&+ P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - 2P_{(\beta)}(Y_1)P_{(\beta)}(Y_2)\\
&- P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2)\\
&- P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma)}(Y_2) - P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma\delta)}(Y_2)\\
&- P_{(\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) - P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\beta\gamma)}(Y_2)\\
&\text{by } (9), (16) \text{ and } (1.5)\\
=&P_{(\beta)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\beta)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big)\\
&+ P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2)\\
&+ P_{(\alpha\beta\gamma)}(Y_2)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big)\\
&+ P_{(\alpha\beta\gamma)}(Y_1)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big)\\
&+ P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma\delta)}(Y_2)\\
&+ P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma)}(Y_2)\\
&+ P_{(\alpha\beta\gamma)}(Y_2)\Big(P_{(\alpha\beta\gamma)}(Y_1) - P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+ P_{(\alpha\beta\gamma)}(Y_1)\Big(P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+ P_{(\alpha\beta\gamma\delta)}(Y_2)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big)\\
&+ P_{(\alpha\beta\gamma\delta)}(Y_1)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
&+ P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
&+ P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
&+ P_{(\beta)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\beta)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
&+ \Big(P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big) \\
&+ \Big(P_{(\alpha\beta\gamma)}(Y_1) + P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_2)\Big(P_{(\alpha\beta\gamma)}(Y_1) - P_{(\beta\gamma\delta)}(Y_1)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_1)\Big(P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma\delta)}(Y_2)\Big) \\
=&P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
&+ P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
&+ P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
&+ P_{(\beta)}(Y_2)P_1\Big(P_{\alpha}(Y_1) - P_{\beta}(Y_1)\Big) + P_{(\beta)}(Y_1)P_2\Big(P_{\alpha}(Y_2) - P_{\beta}(Y_2)\Big) \\
&+ \Big(P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)P_1\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big) \\
&+ \Big(P_{(\alpha\beta\gamma)}(Y_1) + P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)P_2\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_2)P_1\Big(P_{\alpha\beta\gamma}(Y_1) - P_{\beta\gamma\delta}(Y_1)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_1)P_2\Big(P_{\alpha\beta\gamma}(Y_2) - P_{\beta\gamma\delta}(Y_2)\Big) \\
&\text{by } (1.7), (1.9) \text{ and } (1.10).
\end{aligned}
$$

Again by (1.13) and the inductive assumption this term is non-negative, and therefore concludes the proof for $P_{\alpha\beta}(X) \geq P_{\beta\gamma}(X)$. Moreover we have

$$
\begin{aligned}
&P_{\alpha\beta\gamma}(X) - P_{\beta\gamma\delta}(X) \\
=&P_{(\alpha)}(Y_1)P_{(\beta\gamma)}(Y_2) + P_{(\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) + 2P_{(\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) \\
&+ 2P_{(\alpha\beta)}(Y_1)P_{(\beta)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) \\
&+ P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - 3P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) - 3P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) \\
&- P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) - P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
&\text{by } (17) \text{ and } (1.6) \\
=&P_{(\beta\gamma)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\beta\gamma)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
&+ 2P_{(\beta)}(Y_2)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big) + 2P_{(\beta)}(Y_1)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
&+ P_{(\alpha\beta\gamma\delta)}(Y_2)\Big(P_{(\alpha\beta\gamma)}(Y_1) - P_{(\beta\gamma\delta)}(Y_1)\Big) \\
&+ P_{(\alpha\beta\gamma\delta)}(Y_1)\Big(P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma\delta)}(Y_2)\Big) \\
=&P_{(\beta\gamma)}(Y_2)P_1\Big(P_{\alpha}(Y_1) - P_{\beta}(Y_1)\Big) + P_{(\beta\gamma)}(Y_1)P_2\Big(P_{\alpha}(Y_2) - P_{\beta}(Y_2)\Big) \\
&+ 2P_{(\beta)}(Y_2)P_1\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big) + 2P_{(\beta)}(Y_1)P_2\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\beta\gamma\delta)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) \\
&+ P_{(\alpha\beta\gamma\delta)}(Y_2)P_1\Big(P_{\alpha\beta\gamma}(Y_1) - P_{\beta\gamma\delta}(Y_1)\Big) \\
&+ P_{(\alpha\beta\gamma\delta)}(Y_1)P_2\Big(P_{\alpha\beta\gamma}(Y_2) - P_{\beta\gamma\delta}(Y_2)\Big) \\
&\text{by } (1.7), (1.9) \text{ and } (1.10).
\end{aligned}
$$

By (1.13) and the inductive assumption $P_{\alpha\beta\gamma}(X) - P_{\beta\gamma\delta}(X)$ is non-negative, and therefore concludes the proof of the last part of Lemma 1. □

2 Extension to the proof of Theorem 1

First of all we state some equations for $i \in \{1, 2\}$ which helps to show (19).

$$\begin{aligned} 1 &= P_\alpha(Y_i) + P_\beta(Y_i) + P_\gamma(Y_i) + P_\delta(Y_i) + P_{\alpha\beta}(Y_i) + P_{\alpha\gamma}(Y_i) + P_{\alpha\delta}(Y_i) + P_{\beta\gamma}(Y_i) \\ &\quad + P_{\beta\delta}(Y_i) + P_{\gamma\delta}(Y_i) + P_{\alpha\beta\gamma}(Y_i) + P_{\alpha\beta\delta}(Y_i) + P_{\alpha\gamma\delta}(Y_i) + P_{\beta\gamma\delta}(Y_i) + P_{\alpha\beta\gamma\delta}(Y_i) \\ &= P_\alpha(Y_i) + 3P_\beta(Y_i) + 3P_{\alpha\beta}(Y_i) + 3P_{\beta\gamma}(Y_i) + 3P_{\alpha\beta\gamma}(Y_i) + P_{\beta\gamma\delta}(Y_i) + P_{\alpha\beta\gamma\delta}(Y_i) \qquad (2.1) \\ &\quad \text{by } (7), (4), (5). \end{aligned}$$

By (2.1) we have that

$$\begin{aligned} &3P_\beta(Y_i) \\ &= 1 - P_\alpha(Y_i) - 3P_{\alpha\beta}(Y_i) - 3P_{\beta\gamma}(Y_i) - 3P_{\alpha\beta\gamma}(Y_i) - P_{\beta\gamma\delta}(Y_i) - P_{\alpha\beta\gamma\delta}(Y_i) \qquad (2.2) \end{aligned}$$

and

$$\begin{aligned} &P_{(\alpha)}(Y_i) + 3P_{(\beta)}(Y_i) + 3P_{(\alpha\beta)}(Y_i) + 3P_{(\beta\gamma)}(Y_i) + 3P_{(\alpha\beta\gamma)}(Y_i) \\ &\quad + P_{(\beta\gamma\delta)}(Y_i) + P_{(\alpha\beta\gamma\delta)}(Y_i) \\ &= (1-3p_i)P_\alpha(Y_i) + 3p_iP_\beta(Y_i) + 3(1-p_i)P_\beta(Y_i) + 3p_iP_\alpha(Y_i) \\ &\quad + 3(1-2p_i)P_{\alpha\beta}(Y_i) + 6p_iP_{\beta\gamma}(Y_i) + 3(1-2p_i)P_{\beta\gamma}(Y_i) + 6p_iP_{\alpha\beta}(Y_i) \\ &\quad + 3(1-p_i)P_{\alpha\beta\gamma}(Y_i) + 3p_iP_{\beta\gamma\delta}(Y_i) + (1-3p_i)P_{\beta\gamma\delta}(Y_i) \\ &\quad + 3p_iP_{\alpha\beta\gamma}(Y_i) + P_{\alpha\beta\gamma\delta}(Y_i) \\ &\qquad \text{by } (8), (9), (10), (11), (12), (13), (14) \\ &= P_\alpha(Y_i) + 3P_\beta(Y_i) + 3P_{\alpha\beta}(Y_i) + 3P_{\beta\gamma}(Y_i) + 3P_{\alpha\beta\gamma}(Y_i) + P_{\beta\gamma\delta}(Y_i) + P_{\alpha\beta\gamma\delta}(Y_i) \\ &= 1. \qquad (2.3) \end{aligned}$$

Furthermore, the following expressions can be simplified by (6), (2.2) and (2.3).

$$\begin{aligned} &4P_{(\alpha)}(Y_i) + 6P_{(\alpha\beta)}(Y_i) + 4P_{(\alpha\beta\gamma)}(Y_i) + P_{(\alpha\beta\gamma\delta)}(Y_i) \\ =&4\Big(P_{(\alpha)}(Y_i) + \frac{3}{2}P_{(\alpha\beta)}(Y_i) + P_{(\alpha\beta\gamma)}(Y_i) + \frac{1}{4}P_{(\alpha\beta\gamma\delta)}(Y_i)\Big) \\ =&4\Big((1-3p_i)P_\alpha(Y_i) + 3p_iP_\beta(Y_i)) + \frac{3}{2}((1-2p_i)P_{\alpha\beta}(Y_i) + 2p_iP_{\beta\gamma}(Y_i)) \\ &+ (1-p_i)P_{\alpha\beta\gamma}(Y_i) + p_iP_{\beta\gamma\delta}(Y_i) + \frac{1}{4}P_{\alpha\beta\gamma\delta}(Y_i)\Big) \\ =&4\Big((1-3p_i)P_\alpha(Y_i) + p_i\big(1 - P_\alpha(Y_i) - 3P_{\alpha\beta}(Y_i) - 3P_{\beta\gamma}(Y_i) - 3P_{\alpha\beta\gamma}(Y_i) - P_{\beta\gamma\delta}(Y_i) - P_{\alpha\beta\gamma\delta}(Y_i)\big) \\ &+ \frac{3}{2}((1-2p_i)P_{\alpha\beta}(Y_i) + 2p_iP_{\beta\gamma}(Y_i)) + (1-p_i)P_{\alpha\beta\gamma}(Y_i) + p_iP_{\beta\gamma\delta}(Y_i) + \frac{1}{4}P_{\alpha\beta\gamma\delta}(Y_i)\Big) \\ &\text{by } (2.2) \\ =&4\Big((1-3p_i)P_\alpha(Y_i) + p_i - p_iP_\alpha(Y_i) - 3p_iP_{\alpha\beta}(Y_i) - 3p_iP_{\beta\gamma}(Y_i) - 3p_iP_{\alpha\beta\gamma}(Y_i) - p_iP_{\beta\gamma\delta}(Y_i) \\ &- p_iP_{\alpha\beta\gamma\delta}(Y_i) + \frac{3}{2}((1-2p_i)P_{\alpha\beta}(Y_i) + 2p_iP_{\beta\gamma}(Y_i)) + (1-p_i)P_{\alpha\beta\gamma}(Y_i) + p_iP_{\beta\gamma\delta}(Y_i) + \frac{1}{4}P_{\alpha\beta\gamma\delta}(Y_i)\Big) \\ =&4\Big((1-3p_i)P_\alpha(Y_i) + p_i - p_iP_\alpha(Y_i) - 3p_iP_{\alpha\beta}(Y_i) - 3p_iP_{\alpha\beta\gamma}(Y_i) \\ &- p_iP_{\alpha\beta\gamma\delta}(Y_i) + \frac{3}{2}(1-2p_i)P_{\alpha\beta}(Y_i) + (1-p_i)P_{\alpha\beta\gamma}(Y_i) + \frac{1}{4}P_{\alpha\beta\gamma\delta}(Y_i)\Big) \\ =&4\Big(p_i + (1-4p_i)P_\alpha(Y_i) + \frac{3}{2}(1-4p_i)P_{\alpha\beta}(Y_i) + (1-4p_i)P_{\alpha\beta\gamma}(Y_i) + \frac{1}{4}(1-4p_i)P_{\alpha\beta\gamma\delta}(Y_i)\Big) \\ =&4\Big(p_i + (1-4p_i)\Big(P_\alpha(Y_i) + \frac{3}{2}P_{\alpha\beta}(Y_i) + P_{\alpha\beta\gamma}(Y_i) + \frac{1}{4}P_{\alpha\beta\gamma\delta}(Y_i)\Big)\Big) \\ =&4\Big(p_i + (1-4p_i)RA(Y_i)\Big) \qquad (2.4) \\ &\text{by } (6). \end{aligned}$$

Moreover,

$$
\begin{aligned}
&\frac{1}{2}P_{(\alpha)}(Y_i) + \frac{3}{2}P_{(\alpha\beta)}(Y_i) + \frac{3}{2}P_{(\alpha\beta\gamma)}(Y_i) + P_{(\alpha\beta\gamma\delta)}(Y_i) + \frac{3}{2}P_{(\beta)}(Y_i) + P_{(\beta\gamma)}(Y_i) + \frac{1}{4}P_{(\beta\gamma\delta)}(Y_i) \\
&= \frac{1}{2}P_{(\alpha)}(Y_i) + \frac{3}{2}P_{(\beta)}(Y_i) + \frac{3}{2}P_{(\alpha\beta)}(Y_i) + \frac{3}{2}P_{(\beta\gamma)}(Y_i) - \frac{1}{2}P_{(\beta\gamma)}(Y_i) + \frac{3}{2}P_{(\alpha\beta\gamma)}(Y_i) \\
&\quad + \frac{1}{2}P_{(\beta\gamma\delta)}(Y_i) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_i) + \frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_i) + \frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_i) \\
&= \frac{1}{2}\Big(P_{(\alpha)}(Y_i) + 3P_{(\beta)}(Y_i) + 3P_{(\alpha\beta)}(Y_i) + 3P_{(\beta\gamma)}(Y_i) + 3P_{(\alpha\beta\gamma)}(Y_i) + P_{(\beta\gamma\delta)}(Y_i) + P_{(\alpha\beta\gamma\delta)}(Y_i)\Big) \\
&\quad - \frac{1}{2}P_{(\beta\gamma)}(Y_i) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_i) + \frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_i) \\
&= \frac{1}{2} - \frac{1}{2}P_{(\beta\gamma)}(Y_i) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_i) + \frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_i) \qquad (2.5) \\
&\quad \text{by (2.3).}
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
&\frac{3}{2}P_{(\alpha)}(Y_i) + 2P_{(\beta)}(Y_i) + \frac{15}{4}P_{(\alpha\beta)}(Y_i) + \frac{3}{4}P_{(\beta\gamma)}(Y_i) + 3P_{(\alpha\beta\gamma)}(Y_i) + \frac{3}{2}P_{(\alpha\beta\gamma\delta)}(Y_i) \\
&= \frac{2}{3}\Big(P_{(\alpha)}(Y_i) + 3P_{(\beta)}(Y_i) + 3P_{(\alpha\beta)}(Y_i) + 3P_{(\beta\gamma)}(Y_i) + 3P_{(\alpha\beta\gamma)}(Y_i) + P_{(\beta\gamma\delta)}(Y_i) + P_{(\alpha\beta\gamma\delta)}(Y_i)\Big) \\
&\quad + \frac{5}{6}P_{(\alpha)}(Y_i) + \frac{7}{4}P_{(\alpha\beta)}(Y_i) - \frac{5}{4}P_{(\beta\gamma)}(Y_i) + P_{(\alpha\beta\gamma)}(Y_i) - \frac{2}{3}P_{(\beta\gamma\delta)}(Y_i) + \frac{5}{6}P_{(\alpha\beta\gamma\delta)}(Y_i) \\
&= \frac{2}{3} + \frac{5}{6}P_{(\alpha)}(Y_i) + \frac{7}{4}P_{(\alpha\beta)}(Y_i) - \frac{5}{4}P_{(\beta\gamma)}(Y_i) + P_{(\alpha\beta\gamma)}(Y_i) - \frac{2}{3}P_{(\beta\gamma\delta)}(Y_i) + \frac{5}{6}P_{(\alpha\beta\gamma\delta)}(Y_i) \qquad (2.6) \\
&\quad \text{by (2.3),}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{3}{2}P_{(\alpha)}(Y_i) + \frac{3}{4}P_{(\beta)}(Y_i) + 3P_{(\alpha\beta)}(Y_i) + 2P_{(\alpha\beta\gamma)}(Y_i) + P_{(\alpha\beta\gamma\delta)}(Y_i) \\
&= \frac{1}{4}\Big(P_{(\alpha)}(Y_i) + 3P_{(\beta)}(Y_i) + 3P_{(\alpha\beta)}(Y_i) + 3P_{(\beta\gamma)}(Y_i) + 3P_{(\alpha\beta\gamma)}(Y_i) + P_{(\beta\gamma\delta)}(Y_i) + P_{(\alpha\beta\gamma\delta)}(Y_i)\Big) \\
&\quad + \frac{5}{4}P_{(\alpha)}(Y_i) + \frac{9}{4}P_{(\alpha\beta)}(Y_i) - \frac{3}{4}P_{(\beta\gamma)}(Y_i) + \frac{5}{4}P_{(\alpha\beta\gamma)}(Y_i) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_i) + \frac{3}{4}P_{(\alpha\beta\gamma\delta)}(Y_i) \\
&= \frac{1}{4} + \frac{5}{4}P_{(\alpha)}(Y_i) + \frac{9}{4}P_{(\alpha\beta)}(Y_i) - \frac{3}{4}P_{(\beta\gamma)}(Y_i) + \frac{5}{4}P_{(\alpha\beta\gamma)}(Y_i) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_i) + \frac{3}{4}P_{(\alpha\beta\gamma\delta)}(Y_i) \qquad (2.7) \\
&\quad \text{by (2.3).}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
4p_i - 1 &= 4p_i - 4 + 3 + 12p - 12p \\
&= 3 - 12p + 4p_i - 4 + 12(p_i + p_i^{'} - 4p_ip_i^{'}) \\
&\quad \text{by (3)} \\
&= 3(1 - 4p) + 16p_i - 4 + 12p_i^{'} - 48p_ip_i^{'} \\
&= 3(1 - 4p) + 4(1 - 4p_i)(-1 + 3p_i^{'}) \\
&= 3P + 4P_i(-1 + 3p_i^{'}). \qquad (2.8) \\
&\quad \text{by the definition of } P \text{ and } P_i,
\end{aligned}
$$

and

$$
\begin{aligned}
4p_i - 4 + 12p &= 4p_i - 4 + 12(p_i + p_i^{'} - 4p_ip_i^{'}) \\
&\quad \text{by (3)} \\
&= 4(4p_i - 1 + 3p_i^{'} - 12p_ip_i^{'}) \\
&= 4(1 - 4p_i)(-1 + 3p_i^{'}) \\
&= 4P_i(-1 + 3p_i^{'}) \qquad (2.9) \\
&\quad \text{by the definition of } P_i.
\end{aligned}
$$

By using the simplifications stated before we can now rewrite $RA(X)$.

$$
\begin{aligned}
& RA(X) \\
=& P_\alpha(X) + \frac{3}{2}\cdot P_{\alpha\beta}(X) + P_{\alpha\beta\gamma}(X) + \frac{1}{4}\cdot P_{\alpha\beta\gamma\delta}(X) \\
& \text{by } (6) \\
=& P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) + 3P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) + 3P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2) \\
& + 3P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 3P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) + 6P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) \\
& + 3P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 3P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) \\
& + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha)}(Y_2) + \frac{3}{2}\Big(P_{(\alpha)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha)}(Y_2) \\
& + P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) \\
& + P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2)\Big) \\
& + P_{(\alpha)}(Y_1)P_{(\beta\gamma)}(Y_2) + P_{(\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) + 2P_{(\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) \\
& + 2P_{(\alpha\beta)}(Y_1)P_{(\beta)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) \\
& + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + \frac{1}{4}\Big(P_{(\alpha)}(Y_1)P_{(\beta\gamma\delta)}(Y_2) + P_{(\beta\gamma\delta)}(Y_1)P_{(\alpha)}(Y_2) \\
& + 3P_{(\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + 3P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) + 3P_{(\alpha\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
& + 3P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2)\Big) \\
& \text{by } (15), (16), (17), (18) \\
=& P_{(\alpha)}(Y_1)\Big(\frac{1}{2}P_{(\alpha)}(Y_2) + \frac{3}{2}P_{(\alpha\beta)}(Y_2) + \frac{3}{2}P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_2) + \frac{3}{2}P_{(\beta)}(Y_2) + P_{(\beta\gamma)}(Y_2) \\
& \qquad + \frac{1}{4}P_{(\beta\gamma\delta)}(Y_2)\Big) \\
& + P_{(\alpha)}(Y_2)\Big(\frac{1}{2}P_{(\alpha)}(Y_1) + \frac{3}{2}P_{(\alpha\beta)}(Y_1) + \frac{3}{2}P_{(\alpha\beta\gamma)}(Y_1) + P_{(\alpha\beta\gamma\delta)}(Y_1) + \frac{3}{2}P_{(\beta)}(Y_1) + P_{(\beta\gamma)}(Y_1) \\
& \qquad + \frac{1}{4}P_{(\beta\gamma\delta)}(Y_1)\Big) \\
& + P_{(\alpha\beta)}(Y_1)\Big(\frac{3}{2}P_{(\alpha)}(Y_2) + 2P_{(\beta)}(Y_2) + \frac{15}{4}P_{(\alpha\beta)}(Y_2) + \frac{3}{4}P_{(\beta\gamma)}(Y_2) + 3P_{(\alpha\beta\gamma)}(Y_2) + \frac{3}{2}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big) \\
& + P_{(\alpha\beta)}(Y_2)\Big(\frac{3}{2}P_{(\alpha)}(Y_1) + 2P_{(\beta)}(Y_1) + \frac{15}{4}P_{(\alpha\beta)}(Y_1) + \frac{3}{4}P_{(\beta\gamma)}(Y_1) + 3P_{(\alpha\beta\gamma)}(Y_1) + \frac{3}{2}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big) \\
& + P_{(\alpha\beta\gamma)}(Y_1)\Big(\frac{3}{2}P_{(\alpha)}(Y_2) + \frac{3}{4}P_{(\beta)}(Y_2) + 3P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma\delta)}(Y_2)\Big) \\
& + P_{(\alpha\beta\gamma)}(Y_2)\Big(\frac{3}{2}P_{(\alpha)}(Y_1) + \frac{3}{4}P_{(\beta)}(Y_1) + 3P_{(\alpha\beta)}(Y_1) + 2P_{(\alpha\beta\gamma)}(Y_1) + P_{(\alpha\beta\gamma\delta)}(Y_1)\Big) \\
& + \frac{1}{4}P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) \\
=& P_{(\alpha)}(Y_1)\Big(\frac{1}{2} - \frac{1}{2}P_{(\beta\gamma)}(Y_2) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_2) + \frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big) \\
& + P_{(\alpha)}(Y_2)\Big(\frac{1}{2} - \frac{1}{2}P_{(\beta\gamma)}(Y_1) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_1) + \frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big) \\
& + P_{(\alpha\beta)}(Y_1)\Big(\frac{2}{3} + \frac{5}{6}P_{(\alpha)}(Y_2) + \frac{7}{4}P_{(\alpha\beta)}(Y_2) - \frac{5}{4}P_{(\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_2) - \frac{2}{3}P_{(\beta\gamma\delta)}(Y_2) + \frac{5}{6}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big) \\
& + P_{(\alpha\beta)}(Y_2)\Big(\frac{2}{3} + \frac{5}{6}P_{(\alpha)}(Y_1) + \frac{7}{4}P_{(\alpha\beta)}(Y_1) - \frac{5}{4}P_{(\beta\gamma)}(Y_1) + P_{(\alpha\beta\gamma)}(Y_1) - \frac{2}{3}P_{(\beta\gamma\delta)}(Y_1) + \frac{5}{6}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big) \\
& + P_{(\alpha\beta\gamma)}(Y_1)\Big(\frac{1}{4} + \frac{5}{4}P_{(\alpha)}(Y_2) + \frac{9}{4}P_{(\alpha\beta)}(Y_2) - \frac{3}{4}P_{(\beta\gamma)}(Y_2) + \frac{5}{4}P_{(\alpha\beta\gamma)}(Y_2) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_2) + \frac{3}{4}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big) \\
& + P_{(\alpha\beta\gamma)}(Y_2)\Big(\frac{1}{4} + \frac{5}{4}P_{(\alpha)}(Y_1) + \frac{9}{4}P_{(\alpha\beta)}(Y_1) - \frac{3}{4}P_{(\beta\gamma)}(Y_1) + \frac{5}{4}P_{(\alpha\beta\gamma)}(Y_1) - \frac{1}{4}P_{(\beta\gamma\delta)}(Y_1) + \frac{3}{4}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big) \\
& + \frac{1}{4}P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2) \\
& \text{by } (2.5), (2.6), (2.7)
\end{aligned}
$$

$$
\begin{aligned}
=&\Big(\frac{1}{2}P_{(\alpha)}(Y_1)+\frac{5}{4}P_{(\alpha\beta)}(Y_1)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(\frac{1}{2}P_{(\alpha)}(Y_2)+\frac{5}{4}P_{(\alpha\beta)}(Y_2)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_1)+\frac{2}{3}P_{(\alpha\beta)}(Y_1)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_2)+\frac{2}{3}P_{(\alpha\beta)}(Y_2)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+P_{(\alpha)}(Y_1)\Big(\frac{1}{2}+\frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)+P_{(\alpha)}(Y_2)\Big(\frac{1}{2}+\frac{1}{2}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+P_{(\alpha\beta)}(Y_1)\Big(\frac{2}{3}+\frac{1}{3}P_{(\alpha)}(Y_2)+\frac{1}{2}P_{(\alpha\beta)}(Y_2)+\frac{1}{3}P_{(\alpha\beta\gamma)}(Y_2)+\frac{5}{6}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+P_{(\alpha\beta)}(Y_2)\Big(\frac{2}{3}+\frac{1}{3}P_{(\alpha)}(Y_1)+\frac{1}{2}P_{(\alpha\beta)}(Y_1)+\frac{1}{3}P_{(\alpha\beta\gamma)}(Y_1)+\frac{5}{6}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+P_{(\alpha\beta\gamma)}(Y_1)\Big(\frac{1}{4}+P_{(\alpha)}(Y_2)+\frac{3}{2}P_{(\alpha\beta)}(Y_2)+P_{(\alpha\beta\gamma)}(Y_2)+\frac{3}{4}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+P_{(\alpha\beta\gamma)}(Y_2)\Big(\frac{1}{4}+P_{(\alpha)}(Y_1)+\frac{3}{2}P_{(\alpha\beta)}(Y_1)+P_{(\alpha\beta\gamma)}(Y_1)+\frac{3}{4}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{4}P_{(\alpha\beta\gamma\delta)}(Y_1)P_{(\alpha\beta\gamma\delta)}(Y_2)\\
=&\Big(\frac{1}{2}P_{(\alpha)}(Y_1)+\frac{5}{4}P_{(\alpha\beta)}(Y_1)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(\frac{1}{2}P_{(\alpha)}(Y_2)+\frac{5}{4}P_{(\alpha\beta)}(Y_2)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_1)+\frac{2}{3}P_{(\alpha\beta)}(Y_1)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_2)+\frac{2}{3}P_{(\alpha\beta)}(Y_2)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{2}P_{(\alpha)}(Y_1)+\frac{3}{4}P_{(\alpha\beta)}(Y_1)+\frac{1}{2}P_{(\alpha\beta\gamma)}(Y_1)+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_1)\\
&+\frac{1}{2}P_{(\alpha)}(Y_2)+\frac{3}{4}P_{(\alpha\beta)}(Y_2)+\frac{1}{2}P_{(\alpha\beta\gamma)}(Y_2)+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_2)\\
&+P_{(\alpha\beta)}(Y_1)\Big(-\frac{1}{12}+\frac{1}{3}P_{(\alpha)}(Y_2)+\frac{1}{2}P_{(\alpha\beta)}(Y_2)+\frac{1}{3}P_{(\alpha\beta\gamma)}(Y_2)+\frac{1}{12}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+P_{(\alpha\beta)}(Y_2)\Big(-\frac{1}{12}+\frac{1}{3}P_{(\alpha)}(Y_1)+\frac{1}{2}P_{(\alpha\beta)}(Y_1)+\frac{1}{3}P_{(\alpha\beta\gamma)}(Y_1)+\frac{1}{12}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+P_{(\alpha\beta\gamma)}(Y_1)\Big(-\frac{1}{4}+P_{(\alpha)}(Y_2)+\frac{3}{2}P_{(\alpha\beta)}(Y_2)+P_{(\alpha\beta\gamma)}(Y_2)+\frac{1}{4}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+P_{(\alpha\beta\gamma)}(Y_2)\Big(-\frac{1}{4}+P_{(\alpha)}(Y_1)+\frac{3}{2}P_{(\alpha\beta)}(Y_1)+P_{(\alpha\beta\gamma)}(Y_1)+\frac{1}{4}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big(-\frac{1}{8}+\frac{1}{2}P_{(\alpha)}(Y_2)+\frac{3}{4}P_{(\alpha\beta)}(Y_2)+\frac{1}{2}P_{(\alpha\beta\gamma)}(Y_2)+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big(-\frac{1}{8}+\frac{1}{2}P_{(\alpha)}(Y_1)+\frac{3}{4}P_{(\alpha\beta)}(Y_1)+\frac{1}{2}P_{(\alpha\beta\gamma)}(Y_1)+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\Big(\frac{1}{2}P_{(\alpha)}(Y_1)+\frac{5}{4}P_{(\alpha\beta)}(Y_1)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(\frac{1}{2}P_{(\alpha)}(Y_2)+\frac{5}{4}P_{(\alpha\beta)}(Y_2)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_1)+\frac{2}{3}P_{(\alpha\beta)}(Y_1)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_2)+\frac{2}{3}P_{(\alpha\beta)}(Y_2)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{8}\Big(4P_{(\alpha)}(Y_1)+6P_{(\alpha\beta)}(Y_1)+4P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{8}\Big(4P_{(\alpha)}(Y_2)+6P_{(\alpha\beta)}(Y_2)+4P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+\frac{1}{12}P_{(\alpha\beta)}(Y_1)\Big(-1+4P_{(\alpha)}(Y_2)+6P_{(\alpha\beta)}(Y_2)+4P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+\frac{1}{12}P_{(\alpha\beta)}(Y_2)\Big(-1+4P_{(\alpha)}(Y_1)+6P_{(\alpha\beta)}(Y_1)+4P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big(-1+4P_{(\alpha)}(Y_2)+6P_{(\alpha\beta)}(Y_2)+4P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big(-1+4P_{(\alpha)}(Y_1)+6P_{(\alpha\beta)}(Y_1)+4P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big(-1+4P_{(\alpha)}(Y_2)+6P_{(\alpha\beta)}(Y_2)+4P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\\
&+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big(-1+4P_{(\alpha)}(Y_1)+6P_{(\alpha\beta)}(Y_1)+4P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\\
=&\Big(\frac{1}{2}P_{(\alpha)}(Y_1)+\frac{5}{4}P_{(\alpha\beta)}(Y_1)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(\frac{1}{2}P_{(\alpha)}(Y_2)+\frac{5}{4}P_{(\alpha\beta)}(Y_2)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_1)+\frac{2}{3}P_{(\alpha\beta)}(Y_1)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_2)+\frac{2}{3}P_{(\alpha\beta)}(Y_2)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{2}\Big(p_1+(1-4p_1)RA(Y_1)\Big)+\frac{1}{2}\Big(p_2+(1-4p_2)RA(Y_2)\Big)\\
&+\frac{1}{12}P_{(\alpha\beta)}(Y_1)\Big(-1+4\Big(p_2+(1-4p_2)RA(Y_2)\Big)\Big)\\
&+\frac{1}{12}P_{(\alpha\beta)}(Y_2)\Big(-1+4\Big(p_1+(1-4p_1)RA(Y_1)\Big)\Big)\\
&+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big(-1+4\Big(p_2+(1-4p_2)RA(Y_2)\Big)\Big)\\
&+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big(-1+4\Big(p_1+(1-4p_1)RA(Y_1)\Big)\Big)\\
&+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big(-1+4\Big(p_2+(1-4p_2)RA(Y_2)\Big)\Big)\\
&+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big(-1+4\Big(p_1+(1-4p_1)RA(Y_1)\Big)\Big)
\end{aligned}
$$

by (2.4)

$$
\begin{aligned}
=&\Big(\frac{1}{2}P_{(\alpha)}(Y_1)+\frac{5}{4}P_{(\alpha\beta)}(Y_1)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(\frac{1}{2}P_{(\alpha)}(Y_2)+\frac{5}{4}P_{(\alpha\beta)}(Y_2)+\frac{3}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_1)+\frac{2}{3}P_{(\alpha\beta)}(Y_1)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(\frac{1}{4}P_{(\alpha)}(Y_2)+\frac{2}{3}P_{(\alpha\beta)}(Y_2)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+\frac{1}{2}\Big(p_1+P_1RA(Y_1)\Big)+\frac{1}{2}\Big(p_2+P_2RA(Y_2)\Big)\\
&+\Big(\frac{1}{12}P_{(\alpha\beta)}(Y_1)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_1)+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\Big(-1+4\Big(p_2+P_2RA(Y_2)\Big)\Big)\\
&+\Big(\frac{1}{12}P_{(\alpha\beta)}(Y_2)+\frac{1}{4}P_{(\alpha\beta\gamma)}(Y_2)+\frac{1}{8}P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\Big(-1+4\Big(p_1+P_1RA(Y_1)\Big)\Big)\\
&\text{by the definition of } P_1 \text{ and } P_2
\end{aligned}
$$

Then

$$
\begin{aligned}
8D(X)=&8RA(X)-8+24p\\
=&\Big(4P_{(\alpha)}(Y_1)+10P_{(\alpha\beta)}(Y_1)+6P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(4P_{(\alpha)}(Y_2)+10P_{(\alpha\beta)}(Y_2)+6P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(2P_{(\alpha)}(Y_1)+\frac{16}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(2P_{(\alpha)}(Y_2)+\frac{16}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+4p_1+4P_1RA(Y_1)-4+12p+4p_2+4P_2RA(Y_2)-4+12p\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\Big(-1+4p_2+4P_2RA(Y_2)\Big)\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\Big(-1+4p_1+4P_1RA(Y_1)\Big)\\
=&\Big(4P_{(\alpha)}(Y_1)+10P_{(\alpha\beta)}(Y_1)+6P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(4P_{(\alpha)}(Y_2)+10P_{(\alpha\beta)}(Y_2)+6P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(2P_{(\alpha)}(Y_1)+\frac{16}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(2P_{(\alpha)}(Y_2)+\frac{16}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+4P_1RA(Y_1)+4P_1(-1+3p_1^{'})+4P_2RA(Y_2)+4P_2(-1+3p_2^{'})\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\Big(3P+4P_2(-1+3p_2^{'})+4P_2RA(Y_2)\Big)\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\Big(3P+4P_1(-1+3p_1^{'})+4P_1RA(Y_1)\Big)\\
&\text{by } (2.8),(2.9)\\
=&\Big(4P_{(\alpha)}(Y_1)+10P_{(\alpha\beta)}(Y_1)+6P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(4P_{(\alpha)}(Y_2)+10P_{(\alpha\beta)}(Y_2)+6P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(2P_{(\alpha)}(Y_1)+\frac{16}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(2P_{(\alpha)}(Y_2)+\frac{16}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+4P_1(RA(Y_1)-1+3p_1^{'})+4P_2(RA(Y_2)-1+3p_2^{'})\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\Big(3P+4P_2(-1+3p_2^{'}+RA(Y_2))\Big)\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\Big(3P+4P_1(-1+3p_1^{'}+RA(Y_1))\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\Big(4P_{(\alpha)}(Y_1)+10P_{(\alpha\beta)}(Y_1)+6P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta)}(Y_2)-P_{(\beta\gamma)}(Y_2)\Big)\\
&+\Big(4P_{(\alpha)}(Y_2)+10P_{(\alpha\beta)}(Y_2)+6P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta)}(Y_1)-P_{(\beta\gamma)}(Y_1)\Big)\\
&+\Big(2P_{(\alpha)}(Y_1)+\frac{16}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_2)-P_{(\beta\gamma\delta)}(Y_2)\Big)\\
&+\Big(2P_{(\alpha)}(Y_2)+\frac{16}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{(\alpha\beta\gamma)}(Y_1)-P_{(\beta\gamma\delta)}(Y_1)\Big)\\
&+4P_1D_1+4P_2D_2\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\Big(3P+4P_2D_2\Big)\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\Big(3P+4P_1D_1\Big)\\
&\text{by the definition of } D_1, D_2\\
=&\Big(4P_{(\alpha)}(Y_1)+10P_{(\alpha\beta)}(Y_1)+6P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{\alpha\beta}(Y_2)-P_{\beta\gamma}(Y_2)\Big)P_2\\
&+\Big(4P_{(\alpha)}(Y_2)+10P_{(\alpha\beta)}(Y_2)+6P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{\alpha\beta}(Y_1)-P_{\beta\gamma}(Y_1)\Big)P_1\\
&+\Big(2P_{(\alpha)}(Y_1)+\frac{16}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)\Big)\Big(P_{\alpha\beta\gamma}(Y_2)-P_{\beta\gamma\delta}(Y_2)\Big)P_2\\
&+\Big(2P_{(\alpha)}(Y_2)+\frac{16}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)\Big)\Big(P_{\alpha\beta\gamma}(Y_1)-P_{\beta\gamma\delta}(Y_1)\Big)P_1\\
&+4P_1D_1+4P_2D_2\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_1)+2P_{(\alpha\beta\gamma)}(Y_1)+P_{(\alpha\beta\gamma\delta)}(Y_1)\Big)\Big(3P+4P_2D_2\Big)\\
&+\Big(\frac{2}{3}P_{(\alpha\beta)}(Y_2)+2P_{(\alpha\beta\gamma)}(Y_2)+P_{(\alpha\beta\gamma\delta)}(Y_2)\Big)\Big(3P+4P_1D_1\Big)\\
&\text{by } (1.9), (1.10)
\end{aligned}
$$

3 Proof of Lemma 2

To prove Lemma 2 we show that for any rooted binary phylogenetic tree T under a symmetric 3-state substitution model

$$P_\alpha(X) \geq P_\beta(X), \tag{3.1}$$
$$P_{\alpha\beta}(X) \geq P_{\beta\gamma}(X), \tag{3.2}$$

by induction on n. For $n = 2$ the subtrees Y_1 and Y_2 both contain one leaf, which leads to

$$
\begin{aligned}
&P_\alpha(X) = (1-2p_1)(1-2p_2),\\
&P_\beta(X) = p_1p_2,\\
&P_{\alpha\beta}(X) = (1-2p_1)p_2 + p_1(1-2p_2),\\
&P_{\beta\gamma}(X) = 2p_1p_2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
P_\alpha(X) - P_\beta(X) &= (1-2p_1)(1-2p_2) - p_1p_2 = 1 - 2p_1 - 2p_2 + 4p_1p_2 - p_1p_2 = 1 - 2p_1 - 2p_2 + 3p_1p_2\\
&= \underbrace{(1-3p_1)}_{\geq 0}\underbrace{(1-3p_2)}_{\geq 0} + p_1\underbrace{(1-3p_2)}_{\geq 0} + \underbrace{(1-3p_1)}_{\geq 0}p_2 \geq 0 \text{ as } 0 \leq p_1, p_2 \leq \frac{1}{3}.
\end{aligned}
$$

Moreover

$$P_{\alpha\beta}(X) - P_{\beta\gamma}(X) = (1-2p_1)p_2 + p_1(1-2p_2) - 2p_1p_2 = \underbrace{(1-3p_1)}_{\geq 0}p_2 + p_1\underbrace{(1-3p_2)}_{\geq 0} \geq 0 \text{ as } 0 \leq p_1, p_2 \leq \frac{1}{3},$$

which completes the base case of the induction. For the inductive step we first define some recursions similar to (8), (9), (10), (11) and (12):

$$
\begin{aligned}
P_{(\alpha)}(Y_i) &= (1-2p_i)P_\alpha(Y_i) + 2p_iP_\beta(Y_i), && (3.3)\\
P_{(\beta)}(Y_i) &= (1-p_i)P_\beta(Y_i) + p_iP_\alpha(Y_i) = P_{(\gamma)}(Y_i), && (3.4)\\
P_{(\alpha\beta)}(Y_i) &= (1-p_i)P_{\alpha\beta}(Y_i) + p_iP_{\beta\gamma}(Y_i) = P_{(\alpha\gamma)}(Y_i), && (3.5)\\
P_{(\beta\gamma)}(Y_i) &= (1-2p_i)P_{\beta\gamma}(Y_i) + 2p_iP_{\alpha\beta}(Y_i), && (3.6)\\
P_{(\alpha\beta\gamma)}(Y_i) &= P_{\alpha\beta\gamma}(Y_i), && (3.7)
\end{aligned}
$$

With (3.3), (3.4), (3.5), (3.6) and (3.7) we therefore have:

$$
\begin{aligned}
P_\alpha(X) = {} & P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) + 2P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2)\\
& + 2P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
P_\beta(X) = {} & P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\beta)}(Y_2)\\
& + P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) + P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\beta\gamma)}(Y_2)\\
& + P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
P_{\alpha\beta}(X) = {} & P_{(\alpha)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2)\\
& + P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
P_{\beta\gamma}(X) = {} & 2P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2)\\
& + P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \qquad (3.11)
\end{aligned}
$$

Moreover we have that for $i \in \{1,2\}$

$$
\begin{aligned}
P_{(\alpha)}(Y_i) - P_{(\beta)}(Y_i) &= (1-2p_i)P_\alpha(Y_i) + 2p_iP_\beta(Y_i) - (1-p_i)P_\beta(Y_i) - p_iP_\alpha(Y_i)\\
&\qquad \text{by } (3.3), (3.4)\\
&= (1-3p_i)P_\alpha(Y_i) - (1-3p_i)P_\beta(Y_i)\\
&= (1-3p_i)\Big(P_\alpha(Y_i) - P_\beta(Y_i)\Big) \qquad (3.12)
\end{aligned}
$$

and thus

$$
P_{(\alpha)}(Y_i) = P_{(\beta)}(Y_i) + (1-3p_i)\Big(P_\alpha(Y_i) - P_\beta(Y_i)\Big). \qquad (3.13)
$$

In the same manner by (3.5) and (3.6) we can see that

$$
P_{(\alpha\beta)}(Y_i) - P_{(\beta\gamma)}(Y_i) = (1-3p_i)\Big(P_{\alpha\beta}(Y_i) - P_{\beta\gamma}(Y_i)\Big),. \qquad (3.14)
$$

Therefore

$$
P_{(\alpha\beta)}(Y_i) = P_{(\beta\gamma)}(Y_i) + (1-3p_i)\Big(P_{\alpha\beta}(Y_i) - P_{\beta\gamma}(Y_i)\Big). \qquad (3.15)
$$

Additionally we have the following: choose sets A_1, A_2 from $\{\{\alpha\},\{\alpha\beta\}\}$ and B_1, B_2 from $\{\{\beta\},\{\beta\gamma\}\}$ such that for $i \in \{1,2\}$ $|A_i| = |B_i|$, respectively. Then we have that

$$
\begin{aligned}
& P_{(A_1)}(Y_1)P_{(A_2)}(Y_2) - P_{(B_1)}(Y_1)P_{(B_2)}(Y_2)\\
= & \Big(P_{(B_1)}(Y_1) + (1-3p_1)\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\Big)\Big(P_{(B_2)}(Y_2) + (1-3p_2)\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big)\Big)\\
& - P_{(B_1)}(Y_1)P_{(B_2)}(Y_2)\\
& \text{by } (3.13) \text{ or } (3.15)\\
= & P_{(B_1)}(Y_1)P_{(B_2)}(Y_2)\\
& + P_{(B_1)}(Y_1)(1-3p_2)\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big) + P_{(B_2)}(Y_2)(1-3p_1)\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\\
& + (1-3p_1)(1-3p_2)\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big) - P_{(B_1)}(Y_1)P_{(B_2)}(Y_2)\\
= & P_{(B_1)}(Y_1)(1-3p_2)\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big) + P_{(B_2)}(Y_2)(1-3p_1)\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\\
& + (1-3p_1)(1-3p_2)\Big(P_{A_1}(Y_1) - P_{B_1}(Y_1)\Big)\Big(P_{A_2}(Y_2) - P_{B_2}(Y_2)\Big). \qquad (3.16)
\end{aligned}
$$

Now suppose that T has n taxa and that (3.1) and (3.2) are true for all trees having fewer that n taxa. Note that therefore (3.16) is non-negative, since Y_1 and Y_2 contain both fewer than than n taxa. Then

$$\begin{aligned}
&P_\alpha(X) - P_\beta(X) \\
=&P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) + 2P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) + 2P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2) \\
&+ 2P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) + P_{(\alpha)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha)}(Y_2) \\
&- P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\alpha\beta)}(Y_1)P_{(\beta)}(Y_2) \\
&- P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) - P_{(\alpha\beta)}(Y_1) \cdot P_{(\beta\gamma)}(Y_2) \\
&- P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) \\
=&P_{(\alpha)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) \\
&+ P_{(\alpha\beta)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\alpha\beta)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
&+ P_{(\alpha)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
&+ P_{(\alpha\beta)}(Y_1)P_{(\alpha)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta)}(Y_2) \\
&+ P_{(\alpha\beta)}(Y_2)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big) + P_{(\alpha\beta)}(Y_1)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\alpha\beta\gamma)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
=&P_{(\beta)}(Y_1)(1-3p_2)\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big) + P_{(\beta)}(Y_2)(1-3p_1)\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big) \\
&+ (1-3p_1)(1-3p_2)\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big)\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big) \\
&+ P_{(\alpha\beta)}(Y_2)(1-3p_1)\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big) + P_{(\alpha\beta)}(Y_1)(1-3p_2)\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big) \\
&+ P_{(\beta)}(Y_1)(1-3p_2)\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) + P_{(\beta\gamma)}(Y_2)(1-3p_1)\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big) \\
&+ (1-3p_1)(1-3p_2)\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big)\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) \\
&+ P_{(\beta\gamma)}(Y_1)(1-3p_2)\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big) + P_{(\beta)}(Y_2)(1-3p_1)\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big) \\
&+ (1-3p_1)(1-3p_2)\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big)\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big) \\
&+ P_{(\alpha\beta)}(Y_2)(1-3p_1)\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big) + P_{(\alpha\beta)}(Y_1)(1-3p_2)\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_2)(1-3p_1)\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big) + P_{(\alpha\beta\gamma)}(Y_1)(1-3p_2)\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big) \\
&\text{by } (3.12), (3.14), (3.16)
\end{aligned}$$

By the inductive assumption this term is non-negative, and therefore concludes the proof for $P_\alpha(X) \geq P_\beta(X)$. We now proceed with the second part of Lemma 2.

$$\begin{aligned}
&P_{\alpha\beta}(X) - P_{\beta\gamma}(X) \\
=&P_{(\alpha)}(Y_1)P_{(\beta)}(Y_2) + P_{(\beta)}(Y_1)P_{(\alpha)}(Y_2) + P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) \\
&+ P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) + P_{(\alpha\beta\gamma)}(Y_1)P_{(\alpha\beta)}(Y_2) - 2P_{(\beta)}(Y_1)P_{(\beta)}(Y_2) \\
&- P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\alpha\beta\gamma)}(Y_2) - P_{(\alpha\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
=&P_{(\beta)}(Y_2)\Big(P_{(\alpha)}(Y_1) - P_{(\beta)}(Y_1)\Big) + P_{(\beta)}(Y_1)\Big(P_{(\alpha)}(Y_2) - P_{(\beta)}(Y_2)\Big) \\
&+ P_{(\alpha\beta)}(Y_1)P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_1)P_{(\beta\gamma)}(Y_2) \\
&+ P_{(\alpha\beta\gamma)}(Y_2)\Big(P_{(\alpha\beta)}(Y_1) - P_{(\beta\gamma)}(Y_1)\Big) + P_{(\alpha\beta\gamma)}(Y_1)\Big(P_{(\alpha\beta)}(Y_2) - P_{(\beta\gamma)}(Y_2)\Big) \\
=&P_{(\beta)}(Y_2)(1-3p_1)\Big(P_\alpha(Y_1) - P_\beta(Y_1)\Big) + P_{(\beta)}(Y_1)(1-3p_2)\Big(P_\alpha(Y_2) - P_\beta(Y_2)\Big) \\
&+ P_{(\beta\gamma)}(Y_1)(1-3p_2)\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) + P_{(\beta\gamma)}(Y_2)(1-3p_1)\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big) \\
&+ (1-3p_1)(1-3p_2)\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big)\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) \\
&+ P_{(\alpha\beta\gamma)}(Y_2)(1-3p_1)\Big(P_{\alpha\beta}(Y_1) - P_{\beta\gamma}(Y_1)\Big) + P_{(\alpha\beta\gamma)}(Y_1)(1-3p_2)\Big(P_{\alpha\beta}(Y_2) - P_{\beta\gamma}(Y_2)\Big) \\
&\text{by } (3.12), (3.14), (3.16)
\end{aligned}$$

By inductive assumption $P_{\alpha\beta}(X) - P_{\beta\gamma}(X)$ is non-negative, and therefore concludes the proof of the second part of Lemma 2. □